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# Exact analysis of the plane-strain vibrations of thick-walled hollow poroelastic cylinders

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**Abstract**

The paper deals with the plane-strain vibrations of thick walled hollow, composite poroelastic cylinder. The frequency equations of axially and non-axially symmetric vibrations, each for pervious and impervious surfaces are obtained using the analytical model based on Biot's theory of wave propagation in fluid-saturated porous media. In case of axially symmetric vibrations, dilatational and shear modes are uncoupled, while in non-axially symmetric vibrations, dilatational and shear modes are coupled. The plot of frequency versus ratio of wall thickness to inner radius of composite cylinder for different materials is presented, and then discussed. For axially symmetric vibrations, two limiting cases of ratio of wall thickness to inner radius of composite cylinder are considered, i.e., when these ratios are very small and very large. The first limiting case corresponds to modes of thin poroelastic shell and plate, while in the second limiting case, modes of poroelastic solid cylinder is obtained. Thus, the problem of axially symmetric vibrations describes a transition from the case of plate, thereby thin shell to analogous pochammer case of poroelastic solid cylinder. The results of purely elastic solid are shown as a special case. © 2000 Elsevier Science Ltd. All rights reserved.

*Keywords:* Axial symmetry; Non-axial symmetry; Extensional vibrations; Shear vibrations; Pervious surfaces; Impervious surfaces; Poroelastic solid cylinder; Thinshell; Plate

## 1. Introduction

The free vibrations of a solid cylinder of isotropic elastic material is given in Love (1944). Gazis (1958) studied the free vibrations of an infinite thick-walled hollow elastic cylinder with ratio of wall thickness to inner radius.

Using the analytical model based on Biot's theory of wave propagation, the free vibrations of an infinite isotropic poroelastic material is studied by Tajuddin (1978) taking general displacement components of vibratory system following the analysis of Zamanaek (1971). A review of the work based

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on Biot's theory is given by Paria (1963). Tajuddin (1982, 1984) and Tajuddin and Moiz (1984) studied Rayleigh waves in a poroelastic half-space and on curved surfaces, both a convex and a concave surface. A historical formulation of porous media theories is given by Boer and Ehlers (1988). Some problems revealing interesting phenomena which characterise Biot's theory are considered by several authors (Burridge and Vargas, 1979; Jenkins, 1980; Kassir et al., 1989; Jensen et al., 1994; Rajapakse and Senjuntichai, 1995; Chein and Herrmann, 1996).

In the present analysis, the free vibrations of an infinite thick-walled hollow poroelastic cylinder is studied both for axially symmetric and non-axially symmetric vibrations, each for a pervious and an impervious surface. It is assumed that the porous material is homogeneous and isotropic. The frequency equation in each case is derived and discussed. It is intended to describe the transition from the case of a plate thereby thin shell to the analogous case of a poroelastic solid cylinder. Two cases are considered for the ratio of wall thickness ($h$) to inner radius ($r_1$) i.e., $h/r_1$. As a ratio of wall thickness to inner radius tends to zero, modes of an infinite poroelastic plate of thickness equivalent to wall thickness are obtained. All the modes of the thick-walled hollow cylinder approach asymptotically to the analogous modes for a poroelastic solid cylinder of radius ($h$) as the ratio $hr_1^{-1} \to \infty$. Also, it is seen that extensional and shear modes will exist uncoupled in case of axially symmetric vibrations, while it is not true for non-axially symmetric vibrations.

The considered problem is of great practicable interest, particularly in civil engineering, ceramic industry where the frequency of thick-walled hollow poroelastic cylindrical structures play an important role. The investigation can also be applicable in Bio-Mechanics, wherein osseous tissue, bony elements saturated with fluid are approximated by hollow poroelastic cylinder.

## 2. Governing equations

The equations of motion of a poroelastic solid (Biot, 1956) in presence of dissipation ($b$) are

$$N\nabla^2\vec{u} + (A+N)\nabla e + Q\nabla\epsilon = \frac{\partial^2}{\partial t^2}(\rho_{11}\vec{u} + \rho_{12}\vec{U}) + b\frac{\partial}{\partial t}(\vec{u} - \vec{U}),$$

$$Q\nabla e + R\nabla\epsilon = \frac{\partial^2}{\partial t^2}(\rho_{12}\vec{u} + \rho_{22}\vec{U}) - b\frac{\partial}{\partial t}(\vec{u} - \vec{U}), \tag{1}$$

where $\nabla^2$ is the Laplacian, $\vec{u}\ (u,\ v,\ o)$ and $\vec{U}\ (U,\ V,\ O)$ are solid and liquid displacements, $e$ and $\epsilon$ are the dilatations of solid and liquid; $A,\ N,\ Q,\ R$ are poroelastic constants; and $\rho_{ij}$ are mass coefficients following Biot (1956). The relevant solid stresses $\sigma_{ij}$ and liquid pressure $s$ are

$$\sigma_{ij} = 2Ne_{ij} + (Ae + Q\epsilon)\delta_{ij} \quad (i, j = 1, 2, 3),$$

$$s = Qe + R\epsilon. \tag{2}$$

In Eq. (2), $\delta_{ij}$ is the well-known Kronecker delta function.

## 3. Solution of the problem

Let $(r,\ \theta,\ z)$ be cylindrical polar co-ordinates. Consider a thick-walled homogeneous isotropic infinite poroelastic cylinder with inner and outer radii $r_1$ and $r_2$, respectively, whose axis is in the direction of

$z$-axis. Then the solid displacements $\vec{u}$ ($u$, $v$, $o$) which can be readily be evaluated from Eq. (1) representing steady-state harmonic vibrations are

$$u(r, \theta, t) = -\Bigg[C_1\zeta_1\left\{J_{n+1}(\zeta_1 r) - \frac{n}{\zeta_1 r}J_n(\zeta_1 r)\right\} + C_2\zeta_1\left\{Y_{n+1}(\zeta_1 r) - \frac{n}{\zeta_1 r}Y_n(\zeta_1 r)\right\} + C_3\zeta_2\Bigg\{J_{n+1}(\zeta_2 r) - \frac{n}{\zeta_2 r}J_n(\zeta_2 r)\Bigg\} + C_4\zeta_2\left\{Y_{n+1}(\zeta_2 r) - \frac{n}{\zeta_2 r}Y_n(\zeta_2 r)\right\} + A_1\frac{n}{r}J_n(\zeta_3 r) + B_1\frac{n}{r}Y_n(\zeta_3 r)\Bigg]_{\cos}^{\sin}(n\theta)e^{i\omega t},$$

$$v(r, \theta, t) = \Bigg[C_1\frac{n}{r}J_n(\zeta_1 r) + C_2\frac{n}{r}Y_n(\zeta_1 r) + C_3\frac{n}{r}J_n(\zeta_2 r) + C_4\frac{n}{r}Y_n(\zeta_2 r) + A_1\zeta_3\Bigg\{J_{n+1}(\zeta_3 r) - \frac{n}{\zeta_3 r}J_n(\zeta_3 r)\Bigg\} + B_1\zeta_3\left\{Y_{n+1}(\zeta_3 r) - \frac{n}{\zeta_3 r}Y_n(\zeta_3 r)\right\}\Bigg]_{\cos}^{\sin}(n\theta)e^{i\omega t}, \quad (3)$$

[1]

where $\omega$ is the frequency of wave, $n$ is the integer number of waves around the circumference, $C_1$, $C_2$, $C_3$, $C_4$, $A_1$, $B_1$ are all constants, and

$$\zeta_i = \frac{\omega}{V_i}. \quad (i = 1, 2, 3) \quad (4)$$

In Eq. (4), $V_1$, $V_2$ and $V_3$ are dilatational wave velocities of first and second kind and shear wave velocity, respectively (Biot, 1956). By substituting the displacement functions, $u$, $v$ in Eq. (2), the relevant stresses are

$$\sigma_{rr} + s = [C_1 M_{11}(r) + C_2 M_{12}(r) + C_3 M_{13}(r) + C_4 M_{14}(r) + A_1 M_{15}(r) + B_1 M_{16}(r)]_{\cos}^{\sin}(n\theta)e^{i\omega t}, \quad (5)$$

$$\sigma_{r\theta} = [C_1 M_{21}(r) + C_2 M_{22}(r) + C_3 M_{23}(r) + C_4 M_{24}(r) + A_1 M_{25}(r) + B_1 M_{26}(r)]_{\sin}^{\cos}(n\theta)e^{i\omega t}, \quad (6)$$

$$s = [C_1 M_{31}(r) + C_2 M_{32}(r) + C_3 M_{33}(r) + C_4 M_{34}(r)]_{\cos}^{\sin}(n\theta)e^{i\omega t}, \quad (7)$$

$$\frac{\partial s}{\partial r} = [C_1 N_{31}(r) + C_2 N_{32}(r) + C_3 N_{33}(r) + C_4 N_{34}(r)]_{\cos}^{\sin}(n\theta)e^{i\omega t}, \quad (8)$$

where the coefficients $M_{ij}$ and $N_{ij}$ are

$$M_{11}(r) = \left[2N\left(\frac{n(n-1)}{r^2} - \zeta_1^2\right) - (A+Q)\zeta_1^2 + (Q+R)\zeta_1^2\delta_1^2\right]J_n(\zeta_1 r) + \frac{2N\zeta_1}{r}J_{n+1}(\zeta_1 r),$$

$$M_{15}(r) = \frac{2Nn\zeta_3}{r}J_{n+1}(\zeta_3 r) + \frac{2Nn(1-n)}{r^2}J_n(\zeta_3 r),$$

$$M_{21}(r) = \frac{2Nn(n-1)}{r^2}J_n(\zeta_1 r) - \frac{2Nn\zeta_1}{r}J_{n+1}(\zeta_1 r),$$

[1] This notion justifies the existence of either shear vibrations or extensional vibrations when $n = 0$.

$$M_{25}(r) = N\left(\frac{2n(1-n)}{r^2} + \zeta_3^2\right)J_n(\zeta_3 r) - \frac{2N\zeta_3}{r}J_{n+1}(\zeta_3 r),$$

$$M_{31}(r) = (R\delta_1^2 - Q)\zeta_1^2 J_n(\zeta_1 r),$$

$$N_{31}(r) = \left(\frac{R\delta_1^2 - Q}{r}\right)\zeta_1^2 n J_n(\zeta_1 r) + (Q - R\delta_1^2)\zeta_1^3 J_{n+1}(\zeta_1 r),$$

$M_{12}(r)$, $M_{16}(r)$, $M_{22}(r)$, $M_{26}(r)$, $M_{32}(r)$, $N_{32}(r)$ are similar expressions as $M_{11}(r)$, $M_{15}(r)$, $M_{21}(r)$, $M_{25}(r)$, $M_{31}(r)$, $N_{31}(r)$ with $J_n$ and $J_{n+1}$ replaced by $Y_n$ and $\mathrm{Y}_{n+1}$, respectively

$M_{13}(r)$, $M_{23}(r)$, $M_{33}(r)$, $N_{33}(r)$, are similar expressions as $M_{11}(r)$, $M_{21}(r)$, $M_{31}(r)$, $N_{31}(r)$ with $\zeta_1$ and $\delta_1$ replaced by $\zeta_2$ and $\delta_2$, respectively,

$M_{14}(r)$, $M_{24}(r)$, $M_{34}(r)$, $N_{34}(r)$, are similar expressions as $M_{11}(r)$, $M_{21}(r)$, $M_{31}(r)$, $N_{31}(r)$ with $\zeta_1$, $\delta_1$, $J_n$ and $J_{n+1}$ replaced by $\zeta_2$, $\delta_2$, $Y_n$ and $Y_{n+1}$, respectively. (9)

In Eq. (9), $\delta_1^2$ and $\delta_2^2$ are

$$\delta_1^2 = [(RM_{11} - QM_{12}) - V_1^{-2}(PR - Q^2)](RM_{12} - QM_{22})^{-1},$$

$$\delta_2^2 = \text{similar expression as } \delta_1^2 \text{ with } V_1^{-2} \text{ replaced by } V_2^{-2},$$

where $P$ $(=A + 2N)$ is a poroelastic constant, and $M_{11}$, $M_{12}$, $M_{22}$ are

$$M_{11} = \rho_{11} - ib\omega^{-1}, \quad M_{12} = \rho_{12} + ib\omega^{-1}, \quad M_{22} = \rho_{22} - ib\omega^{-1}. \tag{10}$$

## 4. Frequency equation

The boundary conditions for free vibrations in case of a previous surface is

$$\sigma_{rr} + s = 0, \quad \sigma_{r\theta} = 0, \quad s = 0 \quad \text{at} \quad r = r_1 \text{ and } r = r_2, \tag{11}$$

while the boundary conditions for free vibrations in case of an impervious surface is

$$\sigma_{rr} + s = 0, \quad \sigma_{r\theta} = 0, \quad \frac{\partial s}{\partial r} = 0, \text{at} \quad r = r_1 \text{ and } r = r_2. \tag{12}$$

Eqs. (5)–(7) with (11) result in a system of six homogeneous equations in $C_1$, $C_2$, $C_3$, $C_4$, $A_1$ and $B_1$. A nontrivial solution can be obtained if the determinant of the coefficients must vanish. Thus the frequency equation for a pervious surface is

$$\begin{vmatrix} M_{11}(r_1) & M_{12}(r_1) & M_{13}(r_1) & M_{14}(r_1) & M_{15}(r_1) & M_{16}(r_1) \\ M_{21}(r_1) & M_{22}(r_1) & M_{23}(r_1) & M_{24}(r_1) & M_{25}(r_1) & M_{26}(r_1) \\ M_{31}(r_1) & M_{32}(r_1) & M_{33}(r_1) & M_{34}(r_1) & 0 & 0 \\ M_{11}(r_2) & M_{12}(r_2) & M_{13}(r_2) & M_{14}(r_2) & M_{15}(r_2) & M_{16}(r_2) \\ M_{21}(r_2) & M_{22}(r_2) & M_{23}(r_2) & M_{24}(r_2) & M_{25}(r_2) & M_{26}(r_2) \\ M_{31}(r_2) & M_{32}(r_2) & M_{33}(r_2) & M_{34}(r_2) & 0 & 0 \end{vmatrix} = 0. \tag{13}$$

In case of an impervious surface, Eqs. (5), (6), (8) with (12) give the frequency equation

$$\begin{vmatrix} M_{11}(r_1) & M_{12}(r_1) & M_{13}(r_1) & M_{14}(r_1) & M_{15}(r_1) & M_{16}(r_1) \\ M_{21}(r_1) & M_{22}(r_1) & M_{23}(r_1) & M_{24}(r_1) & M_{25}(r_1) & M_{26}(r_1) \\ N_{31}(r_1) & N_{32}(r_1) & N_{33}(r_1) & N_{34}(r_1) & 0 & 0 \\ M_{11}(r_2) & M_{12}(r_2) & M_{13}(r_2) & M_{14}(r_2) & M_{15}(r_2) & M_{16}(r_2) \\ M_{21}(r_2) & M_{22}(r_2) & M_{23}(r_2) & M_{24}(r_2) & M_{25}(r_2) & M_{26}(r_2) \\ N_{31}(r_2) & N_{32}(r_2) & N_{33}(r_2) & N_{34}(r_2) & 0 & 0 \end{vmatrix} = 0. \tag{14}$$

The elements $M_{ij}$ and $N_{ij}$ appearing in Eqs. (13) and (14) are defined in Eq. (9).

By ignoring the liquid effects in frequency Eq. (13), the analogous results of purely elastic solid of Gazis (1958) are obtained as a special case. Due to dissipative nature of the medium, all waves are attenuated. Since that attenuation presents some difficulty in the definition of wave velocity, we will set $b = 0$ in what follows. In addition, it is convenient to introduce non-dimensional variables as follows:

$$a_1 = PH_1^{-1}, \quad a_2 = QH_1^{-1}, \quad a_3 = RH_1^{-1}, \quad a_4 = NH_1^{-1},$$

$$d_1 = \rho_{11}\rho_1^{-1}, \quad d_2 = \rho_{12}\rho_1^{-1}, \quad d_3 = \rho_{22}\rho_1^{-1},$$

$$\tilde{x} = (V_0 V_1^{-1})^2, \quad \tilde{y} = (V_0 V_2^{-1})^2, \quad \tilde{z} = (V_0 V_3^{-1})^2,$$

and

$$b_1 = P_1 H_1^{-1}, \quad b_2 = Q_1 H_1^{-1}, \quad b_3 = R_1 H_1^{-1}, \quad b_4 = N_1 H_1^{-1},$$

$$g_1 = (\rho_{11})_1 \rho_1^{-1}, \quad g_2 = (\rho_{12})_1 \rho_1^{-1}, \quad g_3 = (\rho_{22})_1 \rho_1^{-1}, \quad m = CC_0^{-1}$$

$$\tilde{x}_1 = [V_0 (V_1)_1^{-1}]^2, \quad \tilde{y}_1 = [V_0 (V_2)_1^{-1}]^2, \quad \tilde{z}_1 = [V_0 (V_3)_1^{-1}]^2. \tag{15}$$

In Eq. (15), $C$ is the phase velocity, $C_0$ and $V_0$ are reference velocities ($C_0^2 = \mathrm{N}_1 \rho_1^{-1}$, $V_0^2 = \mathrm{H}_1 \rho_1^{-1}$) then $m$ is non-dimensional phase velocity, and $\rho_1 = (\rho_{11})_1 + 2(\rho_{12})_1 + (\rho_{22})_1$, $H_1 = P_1 + 2Q_1 + R_1$. In all the preceding, and what follows subscript '1' and $(\ )_1$ stand for the quantities related to inner cylinder.

Let

$$g = r_2 r_1^{-1} \text{ so that } \frac{h}{r_1} = g - 1 \text{ and } \Omega = \frac{\omega h}{C_0} = mkh. \tag{16}$$

## 5. Axially symmetric vibrations

In what follows, we set $n = 0$ to consider the motion independent of angular co-ordinate. Then the frequency Eq. (13) for a pervious surface degenerates into the product of two determinants each of second and fourth order respectively, viz,

$$\begin{vmatrix} M_{25}(r_1) & M_{26}(r_1) \\ M_{25}(r_2) & M_{26}(r_2) \end{vmatrix} = 0, \tag{17}$$

and

$$\begin{vmatrix} M_{11}(r_1) & M_{12}(r_1) & M_{13}(r_1) & M_{14}(r_1) \\ M_{31}(r_1) & M_{32}(r_1) & M_{33}(r_1) & M_{34}(r_1) \\ M_{11}(r_2) & M_{12}(r_2) & M_{13}(r_2) & M_{14}(r_2) \\ M_{31}(r_2) & M_{32}(r_2) & M_{33}(r_2) & M_{34}(r_2) \end{vmatrix} = 0, \tag{18}$$

or when written in full

$$(2l_l l_2 m_1 m_2 - l_2^2 m_1^2 - l_1^2 m_2^2)$$

$$k_1(\zeta_2)k_1(\zeta_1) + (l_2 n_1 m_1 m_2 - l_1 n_1 m_2^2)k_1(\zeta_2)[r_2^{-1}k_3(\zeta_1) + r_1^{-1}k_2(\zeta_1)] + (l_1 n_2 m_1 m_2 - l_2 n_2 m_1^2)k_1(\zeta_1)$$

$$[r_2^{-1}k_3(\zeta_2) + r_1^{-1}k_2(\zeta_2)] + \frac{m_1 m_2 n_1 n_2}{r_1 r_2}[k_3(\zeta_2)k_2(\zeta_1) + k_3(\zeta_1)k_2(\zeta_2)$$

$$+k_5(\zeta_1)k_6(\zeta_2) + k_5(\zeta_2)k_6(\zeta_1)] - \frac{n_2^2 m_1^2}{r_1 r_2}k_1(\zeta_1)k_4(\zeta_2) + \frac{n_1^2 m_2^2}{r_1 r_2}k_1(\zeta_2)k_4(\zeta_1) = 0. \tag{19}$$

In Eq. (19), the elements appearing are

$$k_1(\zeta) = J_0(\zeta r_1)Y_0(\zeta r_2) - J_0(\zeta r_2)Y_0(\zeta r_1),$$

$$k_2(\zeta) = J_1(\zeta r_1)Y_0(\zeta r_2) - J_0(\zeta r_2)Y_1(\zeta r_1),$$

$$k_3(\zeta) = J_0(\zeta r_1)Y_1(\zeta r_2) - J_1(\zeta r_2)Y_0(\zeta r_1),$$

$$k_4(\zeta) = J_1(\zeta r_1)Y_1(\zeta r_2) - J_1(\zeta r_2)Y_1(\zeta r_1),$$

$$k_5(\zeta) = J_1(\zeta r_1)Y_0(\zeta r_1) - J_0(\zeta r_1)Y_1(\zeta r_1),$$

$$k_6(\zeta) = J_1(\zeta r_2)Y_0(\zeta r_2) - J_0(\zeta r_2)Y_1(\zeta r_2),$$

$$l_1 = \{(Q+R)\delta_1^2 - (P+Q)\}\zeta_1^2, \quad m_1 = (Q - R\delta_1^2)\zeta_1^2,$$

$$n_1 = 2N\zeta_1, \quad n_2 = 2N\zeta_2,$$

$$l_2, \quad m_2 = \text{similar expressions as } l_1 \text{ and } m_1 \text{ with } \zeta_1^2 \text{ and } \delta_1^2 \text{ replaced by } \zeta_2^2 \text{ and } \delta_2^2\text{, respectively.} \tag{20}$$

Similarly, it can be seen that the frequency Eq. (14) for an impervious surface yields the product of two determinants, each of second and fourth order, respectively. The second order determinant is the same as in Eq. (17), while the fourth order determinant is

$$\begin{vmatrix} M_{11}(r_1) & M_{12}(r_1) & M_{13}(r_1) & M_{14}(r_1) \\ N_{31}(r_1) & N_{32}(r_1) & N_{33}(r_1) & N_{34}(r_1) \\ M_{11}(r_2) & M_{12}(r_2) & M_{13}(r_2) & M_{14}(r_2) \\ N_{31}(r_2) & N_{32}(r_2) & N_{33}(r_2) & N_{34}(r_2) \end{vmatrix} = 0, \tag{21}$$

or when written in full

$$l_1 l_2 m_1 m_2 \zeta_1 \zeta_2 \{k_6(\zeta_2)k_5(\zeta_1) + k_2(\zeta_2)k_3(\zeta_1) + k_2(\zeta_1)k_3(\zeta_2) + k_6(\zeta_1)k_5(\zeta_2)\}$$

$$+ l_1 n_2 m_1 m_2 \zeta_1 \zeta_2 k_4(\zeta_2)\{r_2^{-1}k_3(\zeta_1) + r_1^{-1}k_2(\zeta_1)\} + l_2 n_1 m_1 m_2 \zeta_1 \zeta_2$$

$$\{r_1^{-1}k_2(\zeta_2) + r_2^{-1}k_3(\zeta_2)\}k_4(\zeta_1) + \frac{2n_1 n_2 m_1 m_2 \zeta_1 \zeta_2}{r_1 r_2} k_4(\zeta_1)k_4(\zeta_2)$$

$$-l_2^2 m_1^2 \zeta_1^2 k_4(\zeta_1)k_1(\zeta_2) - l_1^2 m_2^2 \zeta_2^2 k_4(\zeta_2)k_1(\zeta_1) - l_1 n_1 m_2^2 \zeta_2^2$$

$$\{r_2^{-1}k_3(\zeta_1) + r_1^{-1}k_2(\zeta_1)\}k_4(\zeta_2) - l_2 n_2 m_1^2 \zeta_1^2 \{r_2^{-1}k_3(\zeta_2) + r_1^{-1}k_2(\zeta_2)\}k_4(\zeta_2)$$

$$-\frac{n_1^2 m_2^2 \zeta_2^2}{r_1 r_2} k_4(\zeta_2)k_4(\zeta_1) - \frac{n_2^2 m_1^2 \zeta_1^2}{r_1 r_2} k_4(\zeta_1)k_4(\zeta_2) = 0. \tag{22}$$

In Eq. (22), $k_i(\zeta)$, $(i = 1, 2, \ldots 6)$ $l_1$, $m_1$, $n_1$, $l_2$, $m_2$, $n_2$ are defined in Eq. (20).

It can be ascertained that Eq. (17) corresponds to zero dilatational potential, and thus gives the frequency equation of axially symmetric shear modes. Similarly, Eqs. (18) and (21) correspond to zero equivoluminal potential, and give the frequency equations of axially symmetric extensional modes of pervious and impervious surfaces, respectively. It is, therefore, concluded that the shear and extensional modes can exist uncoupled in case of axially symmetric vibration. In addition Eqs. (18) and (21) give the distinct frequencies for pervious and an impervious surfaces, while Eq. (17) is independent of nature of surface. Now, we shall discuss the shear vibrations and extensional vibrations independently.

### *5.1. Shear vibrations*

By substituting $M_{ij}(r)$ from Eq. (9) into Eq. (17), and using recurrence relations (Watson, 1962) the frequency equation simplifies to

$$J_2(\zeta_3 r_1)Y_2(\zeta_3 r_2) - J_2(\zeta_3 r_2)Y_2(\zeta_3 r_1) = 0. \tag{23}$$

In absence of dissipation, using non-dimensional quantities defined in Eqs. (15) with (16) into Eq. (23), one obtains

$$J_2[(\tilde{z}_1 b_4)^{1/2}\Omega/(g-1)]Y_2[(\tilde{z}_1 b_4)^{1/2}g\Omega/(g-1)]$$

$$-J_2[(\tilde{z}_1 b_4)^{1/2}g\Omega/(g-1)]Y_2[(\tilde{z}_1 b_4)^{1/2}\Omega/(g-1)] = 0. \tag{24}$$

Clearly, we see that Eq. (24) gives a relation between the ratio of wall thickness to inner radius $(hr_1^{-1})$ and dimensionless frequency $(\Omega = \omega h/C_0)$.

The frequency Eq. (23) will be discussed for limiting values of the ratio $hr_1^{-1}$ given as follows:

#### *5.1.1. For thin poroelastic cylindrical shell*

When $hr_1^{-1} \ll 1$, and under the assumption $\zeta_3 h \neq 0$, it is seen that $\zeta_3 r_1 \gg 1$ and $\zeta_3 r_2 \gg 1$ so that the Bessel functions $J_2(x)$ and $Y_2(x)$ can be approximated by the first two terms of its Hankel asymptotic series, which in turn, reduces the frequency Eq. (23) to

$$\left(1-\frac{225}{64\zeta_3^2 r_1 r_2}\right)\sin(\zeta_3 h)-\frac{15\zeta_3 h}{8\zeta_3^2 r_1 r_2}\cos(\zeta_3 h)\approx 0. \tag{25}$$

Eq. (25) is the frequency equation for a thin cylindrical poroelastic shell. For $\zeta_3 r_1$ and $\zeta_3 r_2 \to \infty$ in Eq. (25), the limiting frequencies are given by

$$\sin(\zeta_3 h)=0. \tag{26}$$

Then Eq. (26) gives

$$\omega=\frac{q\pi V_3}{h},\quad q=1,2,3\ \ldots \tag{27}$$

which are frequencies of shear modes of a poroelastic plate of thickness $h$.

Furthermore, near the origin $hr_1^{-1}=0$, and assuming $\zeta_3 h=q\pi+\epsilon^*$ ($\epsilon^*\ll 1$), one obtains from the frequency equation of a thin cylindrical poroelastic shell Eq. (25) to be

$$\epsilon^*\approx(hr_1^{-1})^2\left[\frac{15q\pi}{(8q\pi)^2+210(hr_1^{-1})^2}\right]. \tag{28}$$

Finally, we obtain

$$\omega\approx\frac{q\pi}{h}V_3\left[1+\frac{15}{(8q\pi)^2+210(hr_1^{-1})^2}(hr_1^{-1})^2\right]. \tag{29}$$

#### 5.1.2. For poroelastic solid cylinder

When $hr_1^{-1}\gg 1$, Eq. (23) tends asymptotically to

$$J_2(\zeta_3 h)=0. \tag{30}$$

Eq. (30) is the well-known frequency equation of poroelastic solid cylinder of radius $h$ discussed by Tajuddin and Sarma (1980).

### 5.2. Extensional vibrations

Employing the non-dimensional quantities defined in Eqs. (15) with (16) into Eq. (18), one obtains the frequency equation for a pervious surface to be

$$|C_{ij}|=0,\quad (i,j=1,2,3,4) \tag{31}$$

where the elements of determinant are

$$C_{11}=\{(b_2+b_3)(\delta_1^2)_1-(b_1+b_2)\}\tilde{x}_1 b_4\Omega^2 J_0[(\tilde{x}_1 b_4)^{1/2}\Omega/(g-1)]+2b_4(\tilde{x}_1 b_4)^{1/2}\Omega(g-1)J_1[(\tilde{x}_1 b_4)^{1/2}\Omega/(g-1)]$$

$$C_{21}=\{b_3(\delta_1^2)_1-b_2\}\tilde{x}_1 b_4\Omega^2 J_0[(\tilde{x}_1 b_4)^{1/2}\Omega/(g-1)],$$

$$C_{31} = \{(a_2 + a_3)\delta_1^2 - (a_1 + a_2)\}\tilde{x}b_4\Omega^2 J_0[(\tilde{x}b_4)^{1/2}\Omega g/(g-1)] + 2a_4(\tilde{x}b_4)^{1/2}\Omega\{(g - 1)/g\}J_1[(\tilde{x}b_4)^{1/2}\Omega g/(g-1)],$$

$$C_{41} = \{a_2\delta_1^2 - a_2\}\tilde{x}b_4\Omega^2 J_0[(\tilde{x}b_4)^{1/2}\Omega g/(g-1)],$$

$C_{12}$, $C_{32}$ = similar expressions as $C_{11}$, $C_{31}$ with $J_0$, $J_1$ replaced by $Y_0$, $Y_1$ respectively,

$C_{13}$, $C_{23}$ = similar expressions as $C_{11}$, $C_{21}$ with $\tilde{x}_1$ and $(\delta_1^2)_1$ replaced by $\tilde{y}_1$ and $(\delta_2^2)_1$, respectively,

$C_{14}$ = similar expressions as $C_{11}$ with $\tilde{x}_1$, $(\delta_1^2)_1$, $J_0$ and $J_1$ replaced by $\tilde{y}_1$, $(\delta_2^2)_1$, $Y_0$ and $Y_1$, respectively,

$C_{22}$, $C_{42}$ = similar expressions as $C_{21}$, $C_{41}$ with $J_0$ replaced by $Y_0$,

$C_{24}$ = similar expressions as $C_{21}$ with $\tilde{x}_1$, $(\delta_1^2)_1$ and $J_0$ replaced by $\tilde{y}_1$, $(\delta_2^2)_1$ and $Y_0$, respectively,

$C_{33}$, $C_{43}$ = similar expression as $C_{31}$ with $\tilde{x}$ and $\delta_1^2$ replaced by $\tilde{y}$ and $\delta_2^2$, respectively,

$C_{34}$ = similar expression as $C_{31}$ with $\tilde{x}$, $\delta_1^2$, $J_0$ and $J_1$ replaced by $\tilde{y}$, $\delta_2^2$, $Y_0$ and $Y_1$, respectively,

$C_{44}$ = similar expression as $C_{41}$ with $\tilde{x}$, $\delta_1^2$ and $J_0$ replaced by $\tilde{y}$, $\delta_2^2$ and $Y_0$, respectively. (32)

Arguing on similar lines, in case of an impervious surface, we obtain the frequency equation to be

$$|D_{ij}| = 0, \quad (i, j = 1, 2, 3, 4) \tag{33}$$

In Eq. (33), the elements $D_{ij}$ are

$$D_{21} = \{b_2 - b_3(\delta_1^2)_1\}(\tilde{x}_1 b_4)^{3/2}\Omega^3 J_1[(\tilde{x}_1 b_4)^{1/2}\Omega/(g-1)],$$

$$D_{41} = \{a_2 - a_3\delta_1^2\}(\tilde{x}b_4)^{3/2}\Omega^3 J_1[(\tilde{x}b_4)^{1/2}\Omega g/(g-1)],$$

$D_{22}$, $D_{42}$ = similar expression as $D_{21}$, $D_{41}$ with $J_1$ replaced by $Y_1$,

$D_{23}$ = similar expression as $D_{21}$ with $\tilde{x}_1$ and $(\delta_1^2)_1$ replaced by $\tilde{y}_1$ and $(\delta_2^2)_1$, respectively,

$D_{24}$ = similar expression as $D_{21}$ with $J_1$, $\tilde{x}_1$, and $(\delta_1^2)_1$ replaced by $Y_1$, $\tilde{y}_1$, $(\delta_2^2)_1$, respectively,

$D_{43}$ = similar expression as $D_{41}$ with $\tilde{x}$ and $\delta_1^2$ replaced by $\tilde{y}$ and $\delta_2^2$, respectively,

$D_{44}$ = similar expression as $D_{41}$ with $\tilde{x}$, $\delta_1^2$ and $J_1$ replaced by $\tilde{y}$, $\delta_2^2$ and $Y_1$, respectively, and

$$D_{11} = C_{11},\ D_{12} = C_{12},\ D_{13} = C_{13},\ D_{14} = C_{14},$$

$$D_{31} = C_{31},\ D_{32} = C_{32},\ D_{33} = C_{33},\ D_{34} = C_{34}. \tag{34}$$

The quantities $C_{ij}$ appearing in Eq. (34) are defined in Eq. (32).

Eqs. (31) and (33) gives a relation between the ratio $hr_1^{-1}$ and nondimensional frequency $\Omega$. For a fixed $hr_1^{-1}$ $(=g-1)$, $mkh = \Omega$ is a constant, hence the plot relating phase velocity ($m$) and wave number ($kh$) is a rectangular hyperbola. The frequency ($\Omega$) is computed for a trial values of $hr_1^{-1}$ and then included the limiting cases so that a transition from plate, thereby shell vibrations to that of poroelastic solid cylinder can be obtained. The two types of material parameters employed for computational work given by Biot (1956) are presented as follows:

| Material | Parameters | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $d_1$ | $d_2$ | $d_3$ | $\tilde{x}$ | $\tilde{y}$ | $\tilde{z}$ | $b_1$ | $b_2$ | $b_3$ | $b_4$ | $g_1$ | $g_2$ | $g_3$ | $\tilde{x}_1$ | $\tilde{y}_1$ | $\tilde{z}_1$ |
| I | 0.74 | 0.037 | 0.186 | 0.01436 | 0.5 | 0 | 0.5 | 2.725 | 0.673 | 34.82 | 0.61 | 0.0425 | 0.305 | 0.034193 | 0.5 | 0 | 0.5 | 1.671 | 0.812 | 14.623 |
| II | 0.61 | 0.042 | 0.305 | 0.03419 | 0.65 | −0.15 | 0.65 | 2.388 | 0.909 | 18.002 | 0.61 | 0.0425 | 0.305 | 0.034193 | 0.666 | 0 | 0.334 | 1.2121 | 0.996 | 19.477 |

The numerical results are presented in Figs. 1 and 2. It is seen that frequency is more for an impervious surface than for a pervious surface. The values for material-II are more than that of material-I. Following Achenbach and Epstein (1967), material-II is acoustically stiffer and material-I is acoustically softer. Accordingly, the corresponding values for material-II are more than that of material-I, which is true physically (Achenbach and Epstein, 1967).

Frequency Eqs. (19) and (22) will be discussed for limiting values of $hr_1^{-1}$ given as follows:

*5.2.1. For thin poroelastic cylindrical shell*

In the limiting case $hr_1^{-1} \ll 1$, the frequency Eq. (19) for a pervious surface, by means of the asymptotic approximations for the Bessel functions (Watson, 1962) yields

$$\sin(\zeta_1 h)\sin(\zeta_2 h) \approx (h/r_1)^2\left[\frac{2E_1}{\zeta_1 h}L_1 + \frac{2E_2}{\zeta_2 h}L_2 + \frac{8E_3}{\zeta_1\zeta_2 h^2}L_3\right]. \tag{35}$$

In Eq. (35), we have

$$E_1 = N(l_3m_4^2 - l_4m_3m_4)E_4^{-1}, \quad E_2 = N(l_4m_3^2 - l_3m_3m_4)E_4^{-1}, \quad E_3 = N^2m_3m_4E_4^{-1},$$

$$E_4 = 2l_3l_4m_3m_4 - l_4^2m_3^2 - l_3^2m_4^2 - 4N^2(m_3^2\zeta_2^{-2} - m_4^2\zeta_1^{-2})r_1^{-2},$$

$$l_3 = l_1\zeta_1^{-2}, \quad l_4 = l_2\zeta_2^{-2}, \quad m_3 = m_1\zeta_1^{-2}, \quad m_4 = m_2\zeta_2^{-2},$$

$$L_1 = \sin(\zeta_2 h)\cos(\zeta_1 h), \quad L_2 = \sin(\zeta_1 h)\cos(\zeta_2 h), \quad L_3 = \cos(\zeta_1 h)\cos(\zeta_2 h) - 1. \tag{36}$$

Eq. (35) is the frequency equation of extensional modes for a thin poroelastic cylindrical shell.

For $h/r_1 \to 0$, the roots of Eq. (35) tend to the roots of

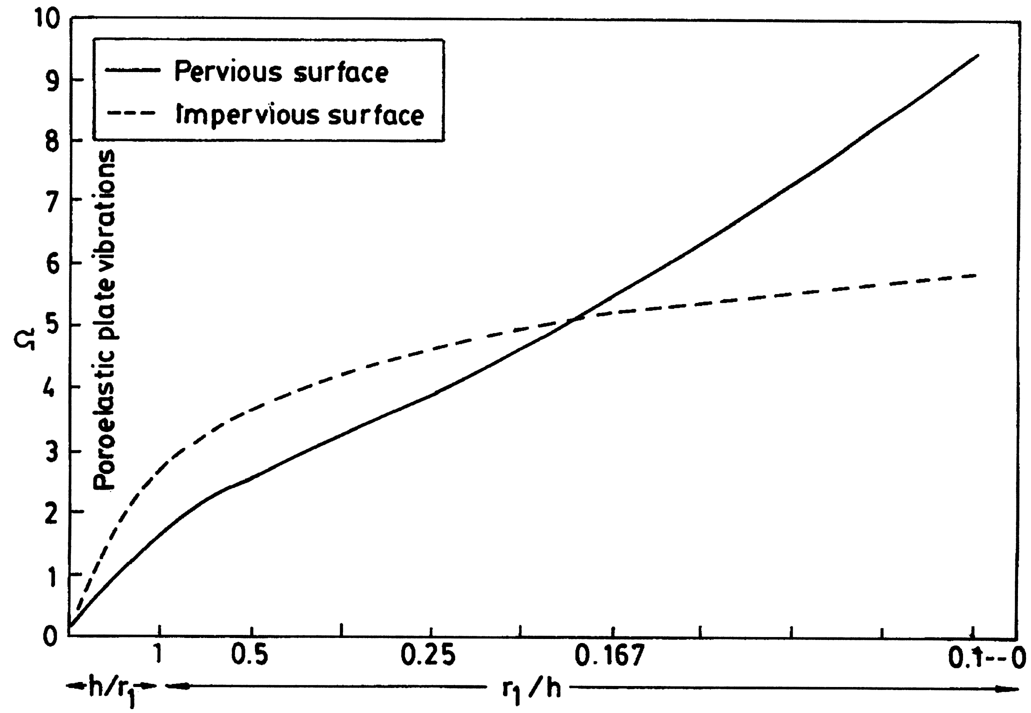


Fig. 1. Axially symmetric vibrations: variation of frequency with $h/r_1$ (composite cylinder—I).

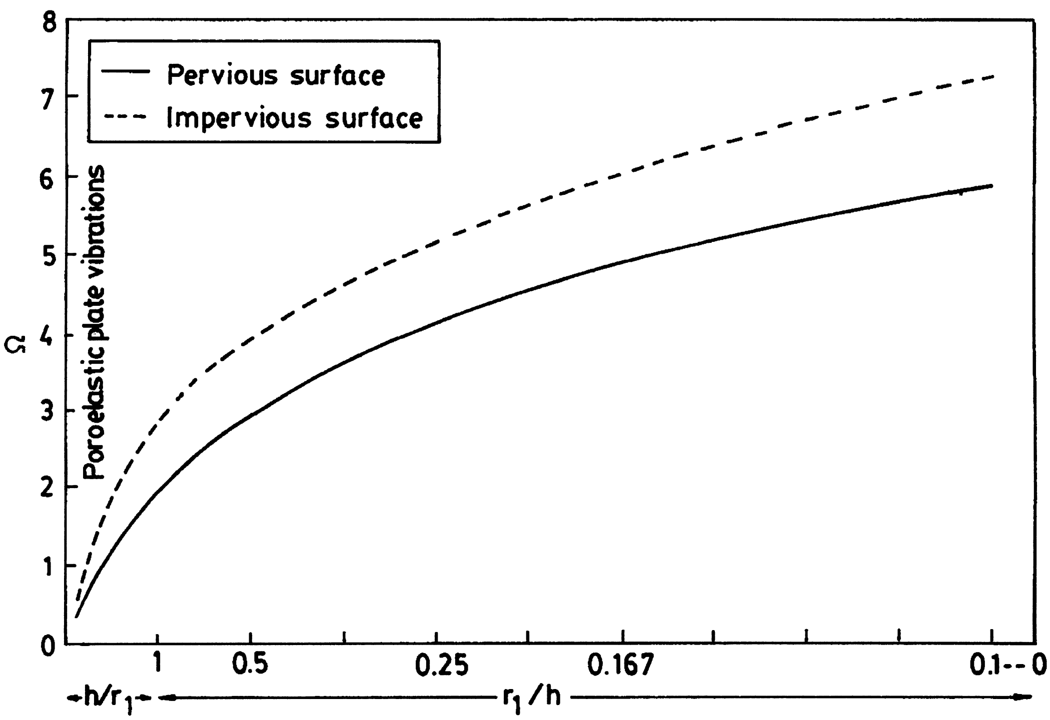


Fig. 2. Axially symmetric vibrations: variation of frequency with $h/r_1$ (composite cylinder—II).

$$\sin(\zeta_1 h)\sin(\zeta_2 h) = 0, \tag{37}$$

provided $\zeta_1 h,\ \zeta_2 h \neq 0$, that is

$$\sin(\zeta_1 h) = 0, \quad \zeta_1 h = p\pi, \quad p = 1, 2, 3\ \ldots$$

hence

$$\omega = \frac{p\pi V_1}{h} \tag{38}$$

or

$$\sin(\zeta_2 h) = 0, \quad \zeta_2 h = q\pi, \quad q = 1, 2, 3\ \ldots$$

hence

$$\omega = \frac{q\pi V_2}{h}. \tag{39}$$

Eqs. (38) and (39) correspond to extensional modes of a poroelastic plate of thickness $h$.

The variation of the frequency-ratios $\zeta_1 h$ and $\zeta_2 h$ in the vicinity of $h/r_1 = 0$ is traced by assuming $\zeta_1 h = p\pi + \epsilon^*$ and/or $\zeta_2 h = q\pi + \epsilon^{*1}$ with $\epsilon^*$, $\epsilon^{*1} \ll 1$. Thus if $\zeta_1 h = p\pi + \epsilon^*$ and $\sin(\zeta_2 h) \neq 0$, Eq. (35) yields

$$\epsilon^* \approx \frac{(hr_1^{-1})^2}{p\pi[1 - (hr_1^{-1})^2(1/p\pi)KE_2\cot(\zeta_2 h)]}\left[E_1 + \frac{KE_3}{p\pi}\left(\frac{\cos(\zeta_2 h) - (-1)^p}{\sin(\zeta_2 h)}\right)\right], \tag{40}$$

and if $\zeta_2 h = q\pi + \epsilon^* 1$ and $\sin(\zeta_1 h) \neq 0$, Eq. (35) yields

$$\epsilon^* 1 \approx \frac{(hr_1^{-1})^2}{q\pi[1 - (hr_1^{-1})^2(1/q\pi)KE_1\cot(\zeta_1 h)]}\left[E_2 + E_3\left(\frac{\cos(\zeta_1 h) - (-1)^q}{\sin(\zeta_1 h)}\right)\right]. \tag{41}$$

In Eqs. (40) and (41), $E_1$, $E_2$, $E_3$ are defined in Eq. (36) and $K$ is given by

$$K = \zeta_1/\zeta_2 = V_2/V_1. \tag{42}$$

The frequency Eq. (22) for an impervious surface by means of asymptotic approximations for the Bessel functions, arguing as before, reduces to

$$\sin(\zeta_1 h)\sin(\zeta_2 h) \approx (h/r_1)^2[E_5 L_1 h^{-1} + E_6 L_2 h^{-1}] + 2E_7 L_3. \tag{43}$$

In Eq. (43), the elements appearing are

$$E_5 = (l_1 n_1 m_2^2 \zeta_2^2 - l_1 n_2 m_1 m_2 \zeta_1 \zeta_2)E_8^{-1},$$

$$E_6 = (l_2 n_2 m_1^2 \zeta_1^2 - l_2 n_1 m_1 m_2 \zeta_1 \zeta_2)E_8^{-1},$$

$$E_7 = l_1 l_2 m_1 m_2 \zeta_1 \zeta_2 E_8^{-1},$$

$$E_8 = 2n_1 n_2 m_1 m_2 \zeta_1 \zeta_2 r_1^2 - n_1^2 m_2^2 \zeta_2^2 r_1^{-2} - n_2^2 m_1^2 \zeta_1^{-2} r_1^{-2} - l_1^2 m_2^2 \zeta_2^2 - l_2^2 m_1^2 \zeta_1^2, \tag{44}$$

where $L_1$, $L_2$, $L_3$ are defined in Eq. (36). The quantities $l_1$, $m_1$, $n_1$, $l_2$, $m_2$, and $n_2$ appearing in Eq. (44) are defined in Eq. (20).

For $hr_1^{-1} \to 0$ in Eq. (43), we obtain

$$\sin(\zeta_1 h)\sin(\zeta_2 h) \approx 2E_7[\cos(\zeta_1 h)\cos(\zeta_2 h) - 1], \tag{45}$$

from which one obtains

$$\omega = \frac{p\pi V_1}{h}, \quad \omega = \frac{q\pi V_2}{h}, \quad (p, q = 1, 2, 3, \ldots) \tag{46}$$

Eq. (46) corresponds to extensional modes of a poroelastic plate of thickness $h$ in case of an impervious surface.

*5.2.2. For poroelastic solid cylinder*

When $hr_1^{-1} \gg 1$, the non-dimensional quantities namely $\tilde{x}_1$, $\tilde{y}_1$, $\tilde{z}_1$, $b_1$, $b_2$, $b_3$, $b_4$, $(\delta_1^2)_1$ and $(\delta_2^2)_1$ tends to $\tilde{x}$, $\tilde{y}$, $\tilde{z}$, $a_1$, $a_2$, $a_3$, $a_4$, $\delta_1^2$ and $\delta_2^2$, respectively. Consequently the non-dimensional frequency Eqs. (31) and (33) for pervious and impervious surfaces asymptotically reduce to

$$C_{31}^* C_{43}^* - C_{33}^* C_{41}^* = 0 \text{ and } C_{31}^* D_{43}^* - C_{33}^* D_{41}^* = 0. \tag{47}$$

In Eq. (47), the elements appearing are

$$C_{31}^* = \{(a_2 + a_3)\delta_1^2 - (a_1 + a_2)\}\Omega J_0[(\tilde{x}a_4)^{1/2}\Omega] + 2a_4^{1/2}\tilde{x}^{-1/2}J_1[(\tilde{x}a_4)^{1/2}\Omega)],$$

$$C_{41}^* = (a_3\delta_1^2 - a_2)J_0(\tilde{x}a_4)^{1/2}\Omega],$$

$$D_{41}^* = (a_2 - a_3\delta_1^2)(\tilde{x})^{1/2}J_1[(\tilde{x}a_4)^{1/2}\Omega],$$

$C_{33}^*$, $C_{43}^*$, $D_{43}^*$ = similar expressions as $C_{31}^*$, $C_{41}^*$, $D_{41}^*$ with $\tilde{x}$ and $\delta_1^2$ replaced by $\tilde{y}$ and $\delta_2^2$, respectively. (48)

Eq. (47) corresponds to frequency equations of poroelastic solid cylinder of radius $h$ for pervious and impervious surfaces, respectively (Tajuddin, 1978).

## 6. Non-axially symmetric vibrations

When the $n$ is non-zero, the dilatational and equivoluminal modes are coupled as indicated by the frequency Eqs. (13) and (14) for both pervious and impervious surfaces, respectively. Employing non-dimensionalisation Eq. (15) into Eq. (13) with Eq. (16), then one obtains the dimensionless frequency equation for a pervious surface given as follows:

$$|A_{ij}| = 0, \quad (i, j = 1, 2, 3, 4, 5, 6) \tag{49}$$

where

$$A_{11} = [2b_4 n(n-1)(g-1)^2 - \{(b_1 + b_2) - (b_2 + b_3)(\delta_1^2)_1\}\tilde{x}_1 b_4 \Omega^2]J_n[(\tilde{x}_1 b_4)^{1/2}\Omega/(g-1)]$$
$$+ \{2b_4(\tilde{x}_1 b_4)^{1/2}\Omega(g-1)\}J_{n+1}[(\tilde{x}_1 b_4)^{1/2}\Omega/(g-1)],$$

$$A_{15} = 2b_4 n(g-1)(\tilde{z}_1 b_4)^{1/2}\Omega J_{n+1}[(\tilde{z}_1 b_4)^{1/2}\Omega/(g-1)] + 2b_4 n(1-n)(g-1)^2 J_n[(\tilde{z}_1 b_4)^{1/2}\Omega/(g-1)],$$

$$A_{21} = 2b_4 n(n-1)(g-1)^2 J_n[(\tilde{x}_1 b_4)^{1/2}\Omega/(g-1)] - 2b_4 n(g-1)(\tilde{x}_1 b_4)^{1/2}\Omega J_{n+1}[(\tilde{x}_1 b_4)^{1/2}\Omega/(g-1)],$$

$$A_{25} = b_4\{2n(1-n)(g-1)^2 + \tilde{z}_1 b_4 \Omega^2\} J_n[(\tilde{z}_1 b_4)^{1/2}\Omega/(g-1)] - 2b_4(g-1)(\tilde{z}_1 b_4)^{1/2}\Omega J_{n+1}[(\tilde{z}_1 b_4)^{1/2}\Omega/(g-1)],$$

$$A_{31} = \{b_3(\delta_1^2)_1 - b_2\}\tilde{x}_1 b_4 \Omega^2 J_n[(\tilde{x}_1 b_4)^{1/2}\Omega/(g-1)],$$

$$A_{41} = \left[2a_4 n(n-1)\left(\frac{g-1}{g}\right)^2 - \{(a_1+a_2)-(a_1+a_3)\delta_1^2\}\tilde{x} b_4 \Omega^2\right] J_n[(\tilde{x} b_4)^{1/2}\Omega g/(g-1)]$$
$$+ \{2a_4(\tilde{x} b_4)^{1/2}\Omega(g-1)/g\} J_{n+1}[(\tilde{x} b_4)^{1/2}\Omega g/(g-1)],$$

$$A_{45} = 2a_4 n(g-1)/g(\tilde{z} b_4)^{1/2}\Omega J_{n+1}[(\tilde{z} b_4)^{1/2}\Omega g/(g-1)] + 2a_4 n(1-n)\left(\frac{g-1}{g}\right)^2 J_n[(\tilde{z} b_4)^{1/2}\Omega g/(g-1)],$$

$$A_{51} = 2a_4 n(n-1)\left(\frac{g-1}{g}\right)^2 J_n[(\tilde{x} b_4)^{1/2}\Omega g/(g-1)] - 2a_4 n(g-1)/g(\tilde{x} b_4)^{1/2}\Omega J_{n+1}[(\tilde{x} b_4)^{1/2}\Omega g/(g-1)],$$

$$A_{55} = a_4\left\{2n(1-n)\left(\frac{g-1}{g}\right)^2 + \tilde{z} b_4 \Omega^2\right\} J_n[(\tilde{z} b_4)^{1/2}\Omega g/(g-1)] - 2a_4(g$$
$$-1)/g(\tilde{z} b_4)^{1/2}\Omega J_{n+1}[(\tilde{z} b_4)^{1/2}\Omega g/(g-1)],$$

$$A_{61} = \{a_3\delta_1^2 - a_2\}\tilde{x} b_4 \Omega^2 J_n[(\tilde{x} b_4)^{1/2}\Omega g/(g-1)],$$

$$A_{35} = A_{36} = A_{65} = A_{66} = 0,$$

$A_{12}, A_{16}, A_{22}, A_{26}, A_{42}, A_{46}, A_{52}, A_{56}$
= similar expression as $A_{11}, A_{15}, A_{21}, A_{25}, A_{41}, A_{45}, A_{51}, A_{55}$ with $J_n$ and $J_{n+1}$
replaced by $Y_n$ and $Y_{n-1}$ respectively,

$A_{13}, A_{33}$ = similar expression as $A_{11}, A_{31}$ with $\tilde{x}_1$ and $(\delta_1^2)_1$ replaced by $\tilde{y}_1$ and $(\delta_2^2)_1$, respectively,

$A_{14}$ = similar expression as $A_{11}$ with $\tilde{x}_1$, $(\delta_1^2)_1$, $J_n$ and $J_{n+1}$ replaced by $\tilde{y}_1$, $(\delta_2^2)_1$, $Y_n$ and $Y_{n+1}$
respectively,

$A_{23}=$ similar expressions as $A_{21}$ with $\tilde{x}_1$ replaced by $\tilde{y}_1$,

$A_{24}=$ similar expression as $A_{21}$, with $J_n$, $J_{n+1}$ and $\tilde{x}_1$ replaced by $Y_n$ and $Y_{n+1}$ and $\tilde{y}_1$, respectively,

$A_{32}$, $A_{62}=$ similar expression as $A_{31}$, $A_{61}$ with $J_n$ replaced by $Y_n$,

$A_{34}=$ similar expression as $A_{31}$ with $\tilde{x}_1$, $(\delta_1^2)_1$ and $J_n$ replaced by $\tilde{y}_1$, $(\delta_2^2)_1$ and $Y_n$, respectively,

$A_{43}$, $A_{63}=$ similar expression as $A_{41}$, $A_{61}$ with $\tilde{x}$ and $\delta_1^2$ replaced by $\tilde{y}$ and $\delta_2^2$, respectively,

$A_{44}=$ similar expression as $A_{41}$ with $\tilde{x}$, $\delta_1^2$, $J_n$ and $J_{n+1}$ replaced by $\tilde{y}$, $\delta_2^2$, $Y_n$ and $Y_{n+1}$, respectively,

$A_{53}=$ similar expression as $A_{51}$ with $\tilde{x}$ replaced by $\tilde{y}$,

$A_{54}=$ similar expression as $A_{51}$ with $J_n$, $J_{n+1}$ and $\tilde{x}$ replaced by $Y_n$, $Y_{n+1}$ and $\tilde{y}$, respectively,

$A_{64}=$ similar expression as $A_{61}$ with $\tilde{x}$, $\delta_1^2$ and $J_n$ replaced by $\tilde{y}$, $\delta_2^2$ and $Y_n$, respectively, (50)

Arguing on similar lines, one obtains the following dimensionless frequency equation for an impervious surface:

$$|B_{ij}|=0. \quad (i,j=1,2,3,4,5,6) \tag{51}$$

In Eq. (51), the elements $B_{ij}$ are

$$B_{31}=\{b_2-b_3(\delta_1^2)_1\}(\tilde{x}_1b_4)^{3/2}\Omega^3J_{n+1}[(\tilde{x}_1b_4)^{1/2}\Omega/(g-1)]-[b_2-b_3(\delta_1^2)_1]\tilde{x}_1b_4\Omega^2(g-1)J_n[(\tilde{x}_1b_4)^{1/2}\Omega/(g-1)],$$

$$B_{61}=\{a_2-a_3\delta_1^2\}(\tilde{x}b_4)^{3/2}\Omega^3J_{n+1}[(\tilde{x}b_4)^{1/2}\Omega g/(g-1)]-\{a_2-a_3\delta_1^2\}\tilde{x}b_4\Omega^2(g-1)/gJ_n[(\tilde{x}b_4)^{1/2}\Omega g/(g-1)],$$

$B_{32}$, $B_{62}=$ similar expression as $B_{31}$, $B_{61}$ with $J_n$ and $J_{n+1}$ replaced by $Y_n$ and $Y_{n+1}$, respectively,

$B_{33}=$ similar expression as $B_{31}$ with $(\delta_1^2)_1$ and $\tilde{x}_1$ replaced by $(\delta_2^2)_1$ and $\tilde{y}_1$, respectively,

$B_{34}=$ similar expression as $B_{31}$ with $J_n$, $J_{n+1}$, $(\delta_1^2)_1$ and $\tilde{x}_1$ replaced by $Y_n$, $Y_{n+1}$, $(\delta_2^2)_1$ and $\tilde{y}_1$, respectively,

$B_{63}=$ similar expression as $B_{61}$ with $\delta_1^2$ and $\tilde{x}$ replaced by $\delta_2^2$ and $\tilde{y}$, respectively,

$B_{64}=$ similar expression as $B_{61}$ with $J_n$, $J_{n+1}$, $\delta_1^2$ and $\tilde{x}$ replaced by $Y_n$, $Y_{n+1}$, $\delta_2^2$ and $\tilde{y}$, respectively,

$$B_{11}=A_{11},\quad B_{12}=A_{12},\quad B_{13}=A_{13},\quad B_{14}=A_{14},\quad B_{15}=A_{15},\quad B_{16}=A_{16},$$

$$B_{21}=A_{21},\quad B_{22}=A_{22},\quad B_{23}=A_{23},\quad B_{24}=A_{24},\quad B_{25}=A_{25},\quad B_{26}=A_{26},$$

$$B_{41}=A_{41},\quad B_{42}=A_{42},\quad B_{43}=A_{43},\quad B_{44}=A_{44},\quad B_{45}=A_{45},\quad B_{46}=A_{46},$$

$$B_{51}=A_{51},\quad B_{52}=A_{52},\quad B_{53}=A_{53},\quad B_{54}=A_{54},\quad B_{55}=A_{55},\quad B_{56}=A_{56},$$

$$B_{35}=B_{36}=B_{65}=B_{66}=0. \tag{52}$$

In Eq. (52), $A_{ij}$ is defined in Eq. (50).

Eqs. (49) and (51) constitute a relation between $hr_1^{-1}$ and $\Omega$. Non-dimensional frequency $\Omega$ versus $hr_1^{-1}$ is computed for two materials whose material parameters have been defined in Section 5.2. The obtained results are presented in Figs. 3 and 4. It is found that the frequency for an impervious surface is less than that of a pervious surface, and the same is less for material-II than for material-I. A case of dissipative medium can also be considered as in Biot (1956) and Tajuddin and Sharma (1980) which needs further a massive amount of detailed analysis. We shall discuss such behaviour later.

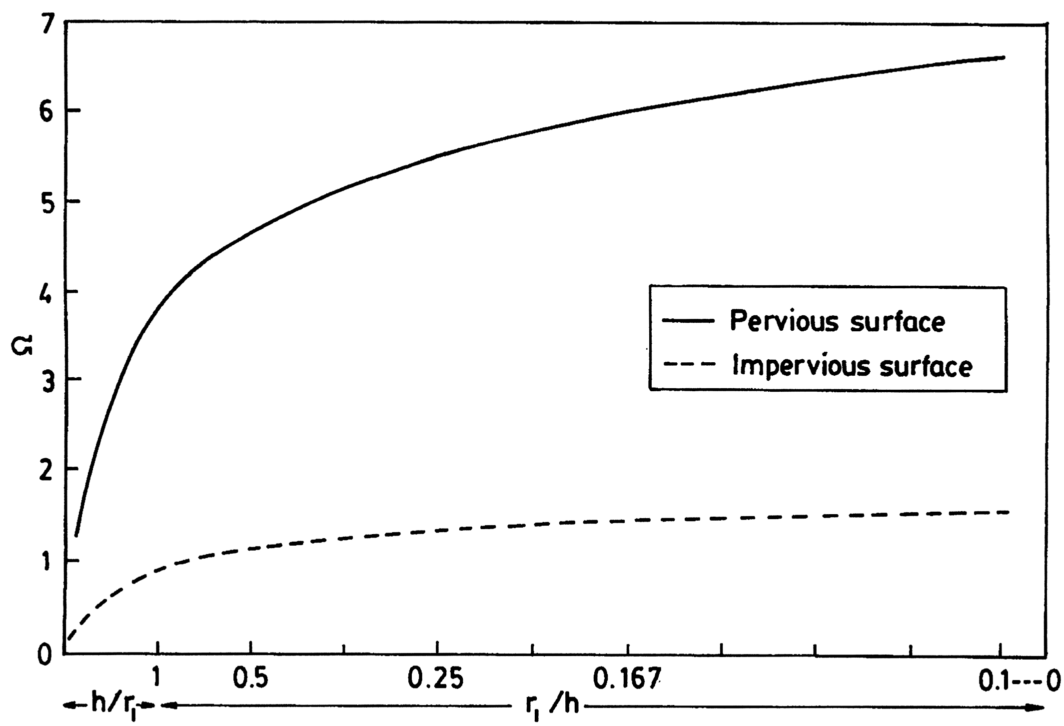


Fig. 3. Non-axially symmetric vibrations: variation of frequency with $h/r_1$ (composite cylinder—I).

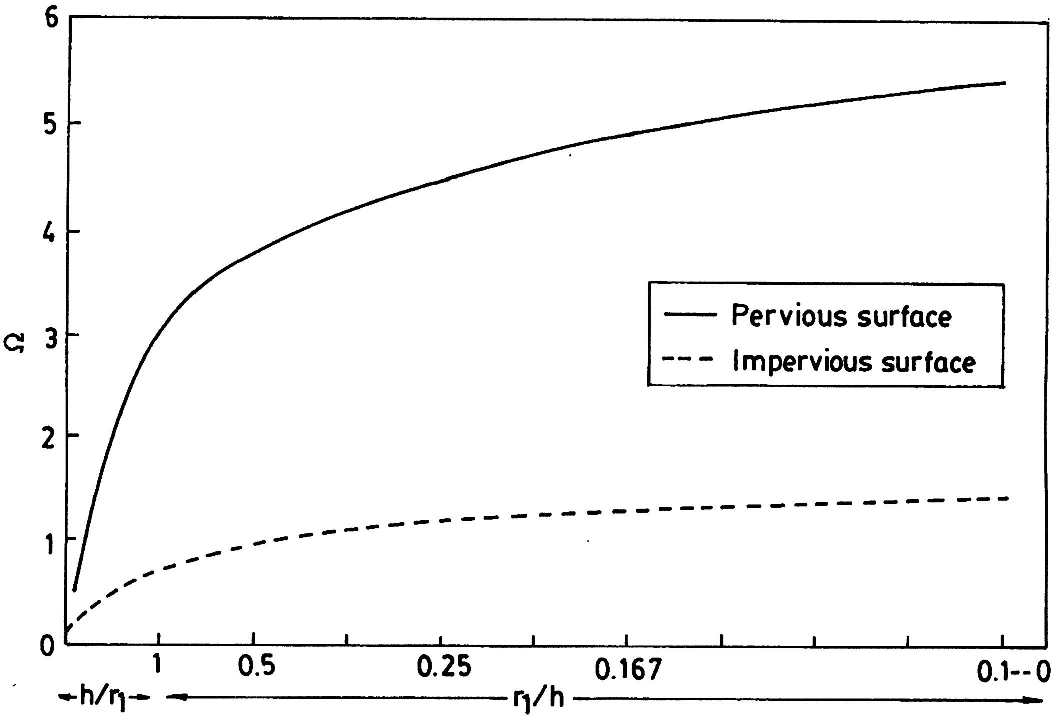


Fig. 4. Non-axially symmetric vibrations: variation of frequency with $h/r_1$ (composite cylinder—II).

## 7. Concluding remarks

The study of plane-strain vibrations of thick-walled hollow poroelastic composite cylinder is made for two types of materials namely material-I and material-II, which are of acoustically softer and acoustically stiffer in characteristic phenomenon. The limiting cases, when $hr_1^{-1} \ll 1$ and $hr_1^{-1} \gg 1$ are discussed, representing thin poroelastic shell and poroelastic solid cylinder, respectively. Thus, the investigation covers the plane-strain vibrations of thick-walled poroelastic hollow cylinders in the entire range of the ratio $hr_1^{-1}$ from zero to $\infty$, so that a transition from plate, thereby shell vibrations to the vibrations of a poroelastic solid cylinder can be seen. The investigation has led to the following conclusions:

1. In case of an axially symmetric vibration
   1.1. the extensional and shear modes exist uncoupled,
   1.2. the shell modes of shear and extensional vibrations approach the modes of a poroelastic plate of thickness $h$, as $hr_1^{-1} \to 0$ for pervious and impervious surfaces,
   1.3. the frequency equation in case of shear vibrations of thick-walled hollow poroelastic cylinder reduces to that of poroelastic solid cylinder of radius $h$ as $hr_1^{-1} \to \infty$,
   1.4. frequency equations in case of extensional vibrations of thick-walled hollow poroelastic cylinder reduces to that of analogous vibrations of poroelastic solid cylinder as $hr_1^{-1} \to \infty$,
   1.5. the frequency for an impervious surface is higher than that of a pervious surface,
   1.6. the frequency for material-II is, in general, higher than that of material-I.

2. In case of non-axially symmetric vibration
   2.1. the extensional and shear modes are coupled.
   2.2. the frequency for an impervious surface is lower than that of a pervious surface,
   2.3. the frequency is less for material-II than for material-I.

3. For both axially and non-axially symmetric vibrations and given $hr_1^{-1}$, the plot of phase velocity versus wave number is a rectangular hyperbola.

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